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An Inequality for Convex Functions

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We prove the following theorem.

THEOREM 1. *Suppose that positive numbers s_i, t_i satisfy $as_i^{-1} + bt_i^{-1} = 1$ $(i = 0, 1, 2)$ for positive constants a_i, b_i and $s_1 \leqslant s_0 \leqslant s_2$. If f, g: $(0, \infty) \to \mathbf{R}$ are convex functions, then*

$$\frac{f(s_0)}{s_0} + \frac{g(t_0)}{t_0} \leqslant \max_{i=1,2} \left(\frac{f(s_i)}{s_i} + \frac{g(t_i)}{t_i}\right).$$

Proof. Choose α_1, α_2 positive such that $\alpha_1 + \alpha_2 = 1$ and $s_0 = \alpha_1 s_1 + \alpha_2 s_2$. Then we have

$$f(s_0) = f(\alpha_1 s_1 + \alpha_2 s_2) \leqslant \alpha_1 f(s_1) + \alpha_2 f(s_2). \tag{1}$$

We choose

$$\beta_1 = \alpha_1 \frac{s_1}{s_0} \frac{t_0}{t_1}, \qquad \beta_2 = \alpha_2 \frac{s_2}{s_0} \frac{t_0}{t_2}$$

and observe that

$$\begin{aligned}\beta_1 + \beta_2 &= \left(\alpha_1 \frac{s_1 - a}{b} + \alpha_2 \frac{s_2 - a}{b}\right) \frac{b}{s_0 - a} \\ &= \frac{\alpha_1 s_1 + \alpha_2 s_2 - a(\alpha_1 + \alpha_2)}{b} \frac{b}{s_0 - a} = 1\end{aligned}$$

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and

$$t_0 = \frac{\alpha_1 s_1 + \alpha_2 s_2}{s_0}\, t_0 = \beta_1 t_1 + \beta_2 t_2.$$

By the convexity of g we have

$$g(t_0) = g(\beta_1 t_1 + \beta_2 t_2) \leqslant \beta_1 g(t_1) + \beta_2 g(t_2). \tag{2}$$

But

$$\frac{\beta_1 t_1}{t_0} = \frac{\alpha_1 s_1}{s_0}, \qquad \frac{\beta_2 t_2}{t_0} = \frac{\alpha_2 s_2}{s_0}, \tag{3}$$

so on combining (1), (2), and (3) we find that

$$\begin{aligned}\frac{f(s_0)}{s_0} + \frac{g(t_0)}{t_0} &\leqslant \alpha_1 \frac{s_1}{s_0} \frac{f(s_1)}{s_1} + \alpha_1 \frac{s_2}{s_0} \frac{f(s_2)}{s_2} + \beta_1 \frac{t_1}{t_0} \frac{g(t_1)}{t_1} + \beta_2 \frac{t_2}{t_0} \frac{g(t_2)}{t_2} \\ &= \alpha_1 \frac{s_1}{s_0} \left(\frac{f(s_1)}{s_1} + \frac{g(t_1)}{t_1} \right) + \alpha_2 \frac{s_2}{s_0} \left(\frac{f(s_2)}{s_2} + \frac{g(t_2)}{t_2} \right) \\ &\leqslant \max_{i=1,2} \left(\frac{f(s_i)}{s_i} + \frac{g(t_i)}{t_i} \right),\end{aligned}$$

since

$$\frac{\alpha_1 s_1}{s_0} + \frac{\alpha_2 s_2}{s_0} = 1.$$

As a special case of our Theorem 1 we obtain inequality (13) of Kemp [1]. In fact we can use our result to obtain the following result which gives an improvement to and generalization of Lemmas 2 and 4 of Brown and Shepp [2].

Let E_i be a non-empty set and L_i a class of non-negative functions $f_i \colon E_i \to \mathbf{R}$ $(i = 1, 2)$. We shall consider functionals $A_i \colon L_i \to \mathbf{R}$ which satisfy the following conditions for $i = 1, 2$.

(a) $f_i \in L_i \Rightarrow A_i(f_i) \geqslant 0$.

(b) $f_i \in L_i$, $\lambda_i > 0 \Rightarrow \lambda_i f_i \in L_i$ and $A_i(\lambda_i f_i) = \lambda_i A_i(f_i)$.

(c) $1 \in L_i$, that is, if $f_i(t) = 1 (\forall t \in E_i)$ then $f_i \in L_i$.

(d) $f_i, g_i \in L_i$ with $f_i(t_i) \geqslant g_i(t_i)$ $(\forall t_i \in E_i) \Rightarrow A_i(f_i) \geqslant A_i(g_i)$.

(e) $A_i(f_i + g_i) \leqslant A_i(f_i) + A_i(g_i)$ $(f_i, g_i \in L_i \Rightarrow f_i + g_i \in L_i)$.

If $f_i^r \in L_i$ $(i = 1, 2)$ for all $r \in (0, \infty)$ then the functions

$$G_i(r) = A_i(f_i^r) \qquad (i = 1, 2)$$

are logarithmically convex on $(0, \infty)$; that is, the functions $\log G_i(r)$ are convex (Pečarić [3]). Hence Theorem 1 gives

THEOREM 2. *Let $f_i\colon L_i \to (0, \infty)$ $(i=1, 2)$ be real functions and let the functionals A_i $(i=1, 2)$ satisfy the five conditions above. Further, let s_i, t_i $(i=0, 1, 2)$ satisfy the conditions of Theorem* 1. *Then*

$$A_1(f_1^{s_0})^{1/s_0}\, A_2(f_2^{t_0})^{1/t_0} \leqslant \max_{i=1,2} A_1(f_1^{s_i})^{1/s_i} A_2(f_2^{t_i})^{1/t_i}.$$

We deduce also the following (cf. [4]).

THEOREM 3. *Let s_i, t_i $(i=0, 1, 2)$ satisfy the conditions of Theorem* 1. *Then*

$$\|f\|_{s_0}\, \|g\|_{t_0} \leqslant \max_{i=1,2} (\|f\|_{s_i}\, \|g\|_{t_i}).$$

THEOREM 4. *Let $x=(x_i)_{i=1}^n$, $u=(u_i)_{i=1}^n$, $y=(y_i)_{i=1}^m$, $v=(u_i)_{i=1}^m$ be sequences of positive numbers, and let s_i, t_i $(i=0, 1, 2)$ satisfy the conditions of Theorem* 1. *Then*

$$S_n^{[s_0]}(x; u)\, S_m^{[t_0]}(y; v) \leqslant \max_{i=1,2} S_n^{[s_i]}(x; u)\, S_m^{[t_i]}(y; v),$$

where

$$S_n^{[t]}(x; u) = \left(\sum_{i=1}^n u_i x_i^t\right)^{1/t}.$$

Moreover, the following generalization of Theorem 1 can be given.

THEOREM 5. *Suppose that positive numbers $s_{i,j}$ $(i=0, 1, 2;\ j=1, ..., n)$ satisfy $s_{i,j} \leqslant s_{0,j} \leqslant s_{2,j}$ $(j=1, ..., n)$ and $a_j s_{i,1}^{-1} + b_j s_{i,j}^{-1} = 1$ $(i=0, 1, 2;\ j=2, ..., n)$ for positive constants a_j, b_j $(j=2, ..., n)$. If $f_j\colon (0, \infty) \to \mathbf{R}$ $(j=1, ..., n)$ are convex functions, then*

$$\sum_{j=1}^n f_j(s_{0,j})/s_{0,j} \leqslant \max_{i=1,2} \left(\sum_{j=1}^n f_j(s_{i,j})/s_{i,j}\right).$$

Proof. Choose $\alpha_{1,1}$, $\alpha_{2,1}$ positive such that $\alpha_{1,1} + \alpha_{2,1} = 1$ and $s_{0,1} = \alpha_{1,1} s_{1,1} + \alpha_{2,1} s_{2,1}$. We have

$$f(s_{0,1}) = f(\alpha_{1,1} s_{1,1} + \alpha_{2,1} s_{2,1}) \leqslant \alpha_{1,1} f(s_{1,1}) + \alpha_{2,1} f(s_{2,1}). \tag{4}$$

We choose

$$\alpha_{1,j} = \alpha_{1,1} \frac{s_{1,1}}{s_{0,1}} \frac{s_{0,j}}{s_{1,j}}, \qquad \alpha_{2,j} = \alpha_{2,1} \frac{s_{2,1}}{s_{0,1}} \frac{s_{0,j}}{s_{2,j}} \qquad (j=2, ..., n)$$

and observe that for $j=2, ..., n$

$$\begin{aligned}
\alpha_{1,j}+\alpha_{2,j} &= \left(\alpha_{1,1}\frac{s_{1,1}}{s_{1,j}}+\alpha_{2,1}\frac{s_{2,1}}{s_{2,j}}\right)\frac{s_{0,j}}{s_{0,1}} \\
&= \left(\alpha_{1,1}\frac{s_{1,1}-a_j}{b_j}+\alpha_{2,1}\frac{s_{2,1}-a_j}{b_j}\right)\frac{b_j}{s_{0,1}-a_j} \\
&= \frac{\alpha_{1,1}s_{1,1}+\alpha_{2,1}s_{2,1}-a_j(\alpha_{1,1}+\alpha_{2,1})}{b_j}\,\frac{b_j}{s_{0,1}-a_j} \\
&= 1
\end{aligned}$$

and

$$s_{0,j}=\frac{\alpha_{1,1}s_{1,1}+\alpha_{2,1}s_{2,1}}{s_{0,1}}\,s_{0,j}=\alpha_{1,j}s_{1,j}+\alpha_{2,j}s_{2,j}.$$

By the convexity of f_j we have

$$f_j(s_{0,j})=f_j(\alpha_{1,j}s_{1,j}+\alpha_{2,j}s_{2,j})\leqslant\alpha_{1,j}f_j(s_{1,j})+\alpha_{2,j}f_j(s_{2,j}). \tag{5}$$

But

$$\left.\begin{aligned}
\alpha_{1,j}\frac{s_{1,j}}{s_{0,j}} &= \alpha_{1,1}\frac{s_{1,1}}{s_{0,1}}, \\
\alpha_{2,j}\frac{s_{2,j}}{s_{0,j}} &= \alpha_{2,1}\frac{s_{2,1}}{s_{0,1}}
\end{aligned}\right.\qquad (j=2, ..., n). \tag{6}$$

On combining (4), (5), and (6) we obtain

$$\begin{aligned}
\sum_{j=1}^{n} f_j(s_{0,j})/s_{0,j} &\leqslant \sum_{j=1}^{n}\,(\alpha_{1,j}f_j(s_{1,j})+\alpha_{2,j}f_j(s_{2,j}))/s_{0,j} \\
&= \sum_{j=1}^{n}\left(\frac{\alpha_{1,j}s_{1,j}}{s_{0,j}}\,\frac{f_j(s_{1,j})}{s_{1,j}}+\frac{\alpha_{2,j}s_{2,j}}{s_{0,j}}\,\frac{f_j(s_{2,j})}{s_{2,j}}\right) \\
&= \frac{\alpha_{1,1}s_{1,1}}{s_{0,j}}\sum_{j=1}^{n}\frac{f_j(s_{1,j})}{s_{1,j}}+\frac{\alpha_{2,1}s_{2,1}}{s_{0,1}}\sum_{j=1}^{n}\frac{f_j(s_{2,j})}{s_{2,j}} \\
&\leqslant \max_{i=1,1}\left(\sum_{j=1}^{n}\frac{f_j(s_{i,j})}{s_{i,j}}\right).
\end{aligned}$$

Analogous generalizations of Theorems 2–4 are easily given.

REFERENCES

1. A. W. KEMP, Certain inequalities involving fractional powers, *J. Austral. Math. Soc. Ser. A* **53** (1992), 131–136.
2. G. BROWN AND L. SHEPP, A convolution inequality, *in* "Contributions to Probability and Statistics. Essays in honour of Ingram Olkin," pp. 51–57, Springer-Verlag, New York, 1989.
3. J. E. PEČARIĆ, Generalization of the power means and their inequalities, *J. Math. Anal. Appl.* **161** (1991), 395–409.
4. C. E. M. PEARCE AND J. E. PEČARIĆ, On two lemmas of G. Brown and L. Shepp, *J. Austral. Math. Soc. Ser. B.*, to appear.